On modelling wave motion in microstructured solids

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Abstract. The Mindlin-type model is used for describing the longitudinal deformation waves in microstructured solids. The evolution equation (one-wave equation) is derived for the hierarchical governing equation (two-wave equation) in the nonlinear case using the asymptotic (reductive perturbation) method. The evolution equation is integrated numerically under harmonic as well as localized initial conditions making use of the pseudospectral method. Analysis of the results demonstrates that the derived evolution equation is able to grasp essential effects of microinertia and elasticity of a microstructure. The influence of these effects can result in the emergence of asymmetric solitary waves.

Key words: nonlinear wave motion, microstructure, hierarchy of waves, evolution equations.

1. INTRODUCTION

In general terms, macrobehaviour of materials depends on properties of the material structure. This is extremely important in contemporary materials science where functionally graded materials, alloys, ceramics, composites, granular materials, etc. are widely used. Proper modelling brings in the scales and hierarchies [6], and the conventional theory of continuous homogeneous media should be considerably enlarged [2,4,11]. The scale dependence involves dispersive effects as shown already in [19]. The hierarchical behaviour in the Whitham sense means that, depending on the ratio of wave characteristics (wavelength) to scales in the material (characteristic scale of a microstructure), the weight of wave operators will be shifted from one to another [21].

One of the ideas to describe the effects of the microstructure is based on Mindlin’s model [12]. This model has recently been extensively studied [2,3], mostly in the 1D setting which explicitly explains the main features of the process. It has been shown that such modelling describes well the influence of the microstructure on dispersion and the existence of hierarchies [2,3]. The model permits, for example, understanding the emergence of solitary waves in microstructured materials, both analytically [9] and numerically [17,18]. In addition, there is a wide area of possible applications in nondestructive testing by solving the corresponding inverse problem for determining the material properties [8,10].

Our final interest is to analyse 2D problems. However, a common approach when solving multidimensional hyperbolic problems is to apply dimensional splitting, i.e., to iterate on 1D problems and to understand the accuracy of possible approximations.

The model equation in the studies mentioned above is in the 1D case a typical hierarchical wave equation with the leading operator of the 2nd order and the higher-order operators (4th, 6th orders) describing the influence of the microstructure [2,3]. This is the two-wave equation, i.e., it describes waves propagating in two directions. The powerful analytic methods [20] show explicitly how in this case evolution equations that govern the propagation of one wave only could be derived. The best example of such an evolution equation is the celebrated Korteweg–de Vries (KdV) equation. The evolution equations may also include hierarchies like in granular materials [7]. If we are interested in wave propagation along a certain coordinate without

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reflection from boundaries, then the concept of evolution
equations is preferable. However, the transformations
from a two-wave model to an evolution equation should
bring over all the essential features that could influence
the velocities or the distortions of the wave profile. It is of
great interest to understand how the hierarchies in basic
Mindlin-type models are reflected in the corresponding
evolution equations and how the solutions describe the
dispersive effects. It must be stressed that once we use
nonlinear models, the balance between nonlinearity and
dispersion is of interest.

The main goals of the present paper are (i) to
derive the evolution equation that governs one-wave
propagation for Mindlin’s model; (ii) to find numerical
solutions to the evolution equation, and (iii) to compare
the results with those of the two-wave equation.

2. BASIC MODEL AND THE EVOLUTION
EQUATION

One-dimensional wave propagation in a microstructured
material has been studied by Engelbrecht et al. [1–3]
on the basis of Mindlin’s model [12], augmented by
nonlinear terms. The motion is described by two scalar
functions, the macrodisplacement \(u(x,t)\) and the micro-
deformation \(\varphi(x,t)\), both depending on the material
coordinate \(x\) and time \(t\). The functions \(u\) and \(\varphi\) are
governed by two coupled partial differential equations of the form

\[
\rho \ddot{u} = \rho u_{xx} + A \varphi_x + \frac{1}{2} N (u^2)_x, \tag{2.1}
\]

\[
I \ddot{\varphi} = C \varphi_{xx} - A u_x - B \varphi + \frac{1}{2} M (\varphi^2)_x, \tag{2.2}
\]

where \(\rho\) and \(I\) denote the macrodensity and the micro-
inertia, respectively, and the constants \(A, A, B, C, N, \) and \(M\) are material parameters specifying the strain
energy function. The last two constants, \(N\) and \(M\), are responsible for nonlinear effects on the macro- and
microscale, respectively.

The main interest is focused on longitudinal waves
modified by the presence of the microstructure. For this
purpose a single partial differential equation is extracted
from the system (2.1), which describes a motion in which
the macrodisplacement prevails and the influence of the
microstructure is retained in a first approximation. The
so-called ‘slaving principle’ is explained in detail in
papers [1–3]. A modified version leading to the same
result is presented in [14]. By keeping the original
parameters, the resulting equation has the form

\[
\rho \ddot{u} = \left( a - \frac{A^2}{B} \right) u_{xx} + \frac{1}{2} N (u^2)_x + \frac{A^2}{B^2} (Iu_x - Cu_{xx})_{xx} + \frac{1}{2} M \frac{A^2}{B^3} (u_{xx})_{xx}, \tag{2.2}
\]

It is an approximate equation extracted from the original
system (2.1) by means of the slaving principle.

Equation (2.2) can still be condensed by introducing
normalized variables and parameters. First, a reference
length \(l\) is chosen. From the original material constants
an inherent length can be extracted, which represents the
size of the microstructure. It is considered to be small
compared to the reference length \(l\) and is introduced by

\[
(\delta l)^2 = \frac{IA^2}{\rho B^2}, \tag{2.3}
\]

where the small number \(\delta \ll 1\) characterizes the small-
ness of the microstructure. In addition, the characteristic
velocities \(c, c_1, c_N,\) and \(c_M\) are defined by

\[
c^2 = \frac{1}{\rho} \left( a - \frac{A^2}{B} \right), \quad c_1^2 = \frac{C}{I}, \quad c_N^2 = \frac{N}{\rho}, \quad c_M^2 = \frac{MA}{Bl} \tag{2.4}
\]

in terms of the basic model parameters and, in the case
of \(c_M^2\), also of the standard length \(l\).

The original variables \(x, t, u\) are finally replaced by
nondimensional variables

\[
X = \frac{x}{l}, \quad T = \frac{ct}{l}, \quad \epsilon U = \frac{u}{l}. \tag{2.5}
\]

The normalization of the displacement uses another
small number \(\epsilon \ll 1\), which emphasizes that the
displacement \(u\) is small compared to the reference
length \(l\). Using the new dimensionless variables, the
governing equation (2.2) assumes the form

\[
U_{TT} = U_{XX} + \frac{1}{2} \epsilon \frac{c_1^2}{c_N^2} (U_N^2)_{XX} + \delta^2 \left( U_{TT} - \frac{c_1^2}{c_N^2} U_{XX} + \frac{1}{2} \epsilon \frac{c_M^2}{c_N^2} U_{XX}^2 \right)_{XX}. \tag{2.6}
\]

If omitting dispersive and nonlinear terms in the
governing equation (2.6), a simple wave equation would
remain, whose general solution would be a left- or right-
going wave of arbitrary shape travelling undisturbed.
Due to the normalization, their speed would be unity.
Let us concentrate on waves propagating to the right.
To include the influence of the additional terms of the
governing equation, we allow the wave profile to change
slowly in time.

In selecting a right-going wave, the solution of the
evolution equation is assumed in the form as suggested in
[13, p. 6]:

\[
U = f(\xi, \tau), \quad \xi = X - T, \quad \tau = \frac{1}{2} \epsilon T, \tag{2.7}
\]

where \(\xi\) and \(\tau\) denote moving space and time
coordinates, respectively. Inserting this ansatz into the
recent form of the governing equation (2.6) and discarding
the higher-order terms, one obtains the equation

\[
-f^2 \xi = \frac{c_1^2}{2 c_N^2} \left( f^2 \xi \right)_\xi + \frac{\delta^2}{\epsilon} \left( f^2 \xi - \frac{c_1^2}{c_N^2} f^2 \xi + \frac{1}{2} \epsilon \frac{c_M^2}{c_N^2} f^2 \xi \right)_{\xi\xi}. \tag{2.8}
\]
Evidently, the influences of dispersion and macro-
nonlinearity, controlled by the two small parameters \( \delta \) and \( \varepsilon \), are balanced only if the quotient \( \delta^2/\varepsilon \) is of the order of unity. Without loss of generality we may assume that \( \varepsilon \) is equal to \( \delta^2 \).

If we denote \( f_\xi = \alpha \), the evolution equation assumes the form

\[
\alpha_\tau + q (\alpha^2)_\xi + z \alpha_{\xi\xi\xi} + w (\alpha^2)_\xi_{\xi\xi} = 0, \tag{2.9}
\]

where the parameters

\[
q = \frac{c_2^2}{2c^2}, \quad z = \frac{c^2 - c_1^2}{c_1^2}, \quad w = \varepsilon \frac{c_1^2}{2c^2} \tag{2.10}
\]

characterize the nonlinearity of macroscale, the dis-
persion, and the nonlinearity of microscale, respect-
ively. Equalizing the micro-nonlinearity parameter \( w \) to zero yields the well-known KdV equation. Thus, compared with the standard KdV equation, equation (2.9) includes an additional complicated term which reflects the nonlinearity on the macroscale.

3. NUMERICAL SIMULATION

The evolution equation (2.9) is solved under harmonic
and localized initial conditions

\[
\alpha(\xi, 0) = \sin \xi, \quad \alpha(\xi, 0) = A_0 \text{sech}^2 \frac{\xi - \xi_0}{\sqrt{12}/A_0}, \tag{3.1}
\]

respectively, where \( A_0 \) is the amplitude, \( \xi_0 \) the initial phase-shift, and \( \sqrt{12}/A_0 \) the width of the initial pulse. For numerical integration the FFT-based pseudospectral method is used and the periodic boundary conditions are applied [5].

The crucial question is the proper choice of parameters because not much is known about the values of physical constants of Mindlin’s model [12]. We choose here the values of parameters comparable with the standard KdV equation which has been studied in detail (see, for example [15,16]). One of the important features of the standard KdV equation is the emergence of a soliton train. The number of solitons in a train depends on the values of \( q \) and \( z \). Widely used values are \( q = 1 \) and \( z = 10^{-2.5} \) [15,16]. Then the soliton train develops at \( \tau \approx 30 \). Another important feature for the KdV equation is the existence of a single stable soliton.

On the basis of the argumentation above, we take
here \( q = 1 \) and vary the other parameters in the following domains: \( 10^{-2.5} \leq z \leq 1 \) and \( 0 \leq w \leq 1 \). The localized initial wave (3.1.2) is the analytical solution for equation (2.9) in the case of \( w = 0 \), i.e., it represents the KdV soliton.

3.1. Localized initial excitation

Janno and Engelbrecht [9] have shown that for the two-
wave equation (2.6) there exists an asymmetric travelling wave solution, i.e., the nonlinearity in microscale leads to asymmetry of the wave profile. Numerical experiments by Salupere et al. [17,18] have demonstrated that in the case of equation (2.6), an initially symmetric localized wave is deformed to an asymmetric wave during propagation. Here we show that the same effect takes place in the case of the evolution equation (2.9).

The evolution of the initial symmetric \( \text{sech}^2 \) pulse can be traced in Fig. 1. It is clear that the shape of the wave is altered during propagation and an oscillating tail is formed. In Fig. 2 the initial wave profile and the altered shape of the wave profile at the end of the integration interval are plotted against \( \xi \). In order to characterize the asymmetry of the last wave profile more explicitly, \( \alpha_\xi \) is plotted against \( \alpha \) in Fig. 3.

\[\text{Fig. 1. Time-slice plot for } z = 10^{-2}, \text{ } w = 10^{-2.5}.\]

\[\text{Fig. 2. The initial (dashed line) and the deformed (solid line) wave profile from Fig. 1 } (z = 10^{-2}, \text{ } w = 10^{-2.5}).\]
In applying localized initial conditions the value of the micro-nonlinearity parameter \( w = 10^{-2.5} \) is chosen quite big compared to the macro-nonlinearity parameter \( q \) and the dispersion parameter \( z \) in order to demonstrate the effect of asymmetry more clearly.

### 3.2. Harmonic initial excitation

It is of interest to start with the case \( w = 0 \) which corresponds to a standard KdV equation. This means that micro-nonlinearity is neglected. As typical of the KdV case, a train of solitons will emerge from a harmonic initial excitation (Fig. 4.) The interaction picture is complicated but solitons preserve their shape and speed over long time intervals. The soliton amplitudes fluctuate in the interval that is dictated by the interaction rules [15,16]. When the micro-nonlinearity is taken into account, the interaction pattern is altered – speeds of solitons are higher than in the KdV case (cf. Figs 4 and 5). Like in the case of localized initial conditions, the emerged solitons (Fig. 6) are asymmetric, as can be observed from the phase plane, i.e., the \( (\alpha, \alpha_\xi) \) plot (Fig. 7). This is a clear sign of the influence of
4. CONCLUDING REMARKS

The evolution equation (2.9) that governs one-wave propagation in microstructured solids according to Mindlin’s model is derived and solved numerically under harmonic and localized initial conditions. Analysis of numerical results demonstrates that (i) for both the governing equation and the evolution equation nonlinearity in microscale leads to asymmetry of the wave profile; and (ii) the stronger the influence of micrononlinearity, the more the solutions of the evolution equation differ from those of the KdV model. In conclusion, the derived evolution equation (2.9) – notwithstanding that it is a simplified model equation compared to the two-wave equation (2.6) – is able to grasp essential effects of microinertia and elasticity of a microstructure. However, we stress that the values of parameters used above are chosen for the comparison with the standard KdV equation in order to demonstrate the influence of the microstructure. Studies with other parameters are in progress. A real challenge is to find an analytical solution to equation (2.9).

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REFERENCES

Lainelevi modelleerimisest mikrostruktuuriga materjalides

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