Proceedings of the Estonian Academy of Sciences,
2010, **59**, 4, 355–363
doi: 10.3176/proc.2010.4.15
Available online at www.eap.ee/proceedings

MATHEMATICS

On the non-Koszulity of ternary partially associative operad

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Received 30 September 2009, accepted 14 April 2010

Abstract. We prove that the operad for ternary partially associative algebras is non Koszul. The aim is to underline the problem of computing the dual operad when we consider quadratic operad for n-ary algebras in particular when n is odd. In fact, the dual operad is generally defined in the graded (differential) operad framework. The result of non-Koszulity extends for other operads for $(2p+1)$-ary partially associative algebras although the operads for $(2p)$-ary partially associative algebras are Koszul.

Key words: n-ary algebras, operads, partial/total associativity, degree d generating operation, Koszulity.

1. INTRODUCTION

We are interested, for any natural number n, in the operad, denoted $p\mathscr{A}ss_0^n$, associated to n-ary algebras whose multiplication μ satisfies

$$\sum_{i=1}^{n}(-1)^{(i-1)(n-1)}\mu(X_1,\cdots,X_{i-1},\mu(X_i,\cdots,X_{i+n-1}),X_{i+n},\cdots,X_{2n-1})=0.$$

Such a multiplication is called n-ary partially associative. These operads were first studied by Gnedbaye [3]. This paper, which resumes my talk given in the meeting of Tartu, concerns more precisely the non-Koszulity of these operads when n is odd equal to 3. It is a part of a larger work co-authored by Martin Markl [8].

When computing the free algebras associated to $(2p+1)$-ary partially associative algebras, using different arguments in [7] and [8], we saw that the cases n even and n odd behaved in a completely different way. Our approach is slightly different from the approach in [3], which contains some misunderstanding of the odd case. In fact, contrary to what we find in this paper, the operads associated to ternary partially associative algebras and other $(2p+1)$-ary partially associative algebras are not Koszul and so the operadic cohomology does not capture deformations.

To study the (non-)Koszulity of $p\mathscr{A}ss_0^n$, we need to define the dual operad, which involves understanding the definition of Ginzburg and Kapranov [2] developed in the case of binary operations in order to extend it to n-ary operations. When we compute the dual of a graded or nongraded operad, we get graded objects which involve suspensions. In particular, if we consider an n-ary multiplication of degree 0 and its associated operad $\mathscr{P}$, an algebra on its dual operad $\mathscr{P}^!$ corresponds to an n-ary multiplication of degree $n-2$. In our case, as n is odd, the multiplication of an algebra on the dual operad is of odd degree, i.e. can be placed in degree 1 and not in degree 0. If we forget this degree we get an operad over algebras with multiplication of even degree which does not correspond to the dual operad in the Ginzburg–Kapranov sense when n is odd.

In the following we consider $\mathbb{K}$ a field of characteristic 0 and the operads that we discuss are generally $\mathbb{K}$-linear operads. All definitions and concepts used refer to [2] and [9].

2. THE OPERAD $p\mathscr{A}ss_0^3$

This section deals with the operad of ternary – i.e. 3-ary – partially associative algebras, that is, algebras defined by a multiplication

$$\mu : A^{\otimes 3} \to A$$

satisfying the relation

$$\mu \circ (\mu \otimes I_2) + \mu \circ (I_1 \otimes \mu \otimes I_1) + \mu \circ (I_2 \otimes \mu) = 0,$$

where $I_j : A^{\otimes j} \to A^{\otimes j}$ is the identity map. We have a classical example of such an algebra when we consider the Hochschild cohomology of an associative algebra [1]. If $\mathscr{C}^k(V,V)$ denotes the space of k-cochains of the Hochschild cohomology of the associative algebra V, the Gerstenhaber product $\circ_{n,m}$ is a linear map

$$\circ_{n,m} : \mathscr{C}^n(V,V) \times \mathscr{C}^m(V,V) \to \mathscr{C}^{n+m-1}(V,V)$$

given by

$$f \circ_{n,m} g(X_1 \otimes \cdots \otimes X_{n+m-1}) = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f(X_1 \otimes \cdots \otimes g(X_i \otimes \cdots \otimes X_{i+m-1}) \otimes \cdots \otimes X_{n+m-1}),$$

if $f \in C^n(V)$ and $g \in C^m(V)$. A ternary partially associative product is a 3-cochain μ satisfying $\mu \circ_{3,3} \mu = 0$.

Recall the basic notions of operad and quadratic operad (see [2]). An *operad* $\mathscr{P}$ consists of a collection $\{\mathscr{P}(n), n \geq 1\}$ of $\mathbb{K}$-vector spaces such that each $\mathscr{P}(n)$ is a Σ_n-module, where Σ_n is the symmetric group on n elements. There is an element $1 \in \mathscr{P}(1)$ called the unit and linear maps

$$\circ_i : \mathscr{P}(n) \times \mathscr{P}(m) \to \mathscr{P}(n+m-1)$$

called comp-i operations satisfying associativity conditions, and the comp-i operations are compatible with the action of the symmetric group.

Recall that a $\mathscr{P}$-algebra is a $\mathbb{K}$ vector space V equipped with a morphism of operads $f : \mathscr{P} \to \mathscr{E}_V$ where $\mathscr{E}_V$ is the operad of endomorphisms of V. Giving a structure of a $\mathscr{P}$-algebra on V is the same as giving a collection of linear maps

$$f_n : \mathscr{P}(n) \otimes V^{\otimes n} \to V$$

satisfying natural associativity, equivariance, and unit conditions.

If E is a right-$\mathbb{K}[\Sigma_2]$-module, we can define an operad, denoted by $\mathscr{F}(E)$ and called the free operad generated by E which is the solution of the following universal problem: for any operad $\mathscr{Q} = \{\mathscr{Q}(n)\}$ and any $\mathbb{K}[\Sigma_2]$-linear morphism $f : E \to \mathscr{Q}(2)$, there exists a unique operad morphism $\hat{f} : \mathscr{F}(E) \to \mathscr{Q}$ which coincides with f on $E = \mathscr{F}(E)(2)$. We have for example

$$\mathscr{F}(E)(3) = (E \otimes E) \otimes_{\Sigma_2} \mathbb{K}[\Sigma_3].$$

If R is a $\mathbb{K}[\Sigma_3]$-submodule of $\mathscr{F}(E)(3)$, it generates the operadic ideal $\mathscr{R} = (R)$ of $\mathscr{F}(E)$. The quadratic operad generated by E with relations R is the operad $\mathscr{P}(\mathbb{K}, E, R) = \{\mathscr{P}(\mathbb{K}, E, R)(n), n \geq 1\}$ with

$$\mathscr{P}(\mathbb{K}, E, R)(n) = \mathscr{F}(E)(n)/\mathscr{R}(n).$$

This notion of quadratic operad is related to binary algebras. In [3] this notion is adapted to n-ary algebras. In this case we consider a generating multiplication μ which is an n-ary multiplication, that is $E =< \mu >$ generated as a $\mathbb{K}[\Sigma_n]$-module. We define the free operad $\mathscr{F}(E)$ generated by E in the same way as in the binary case and the ideal of relations with R is a $\mathbb{K}[\Sigma_{2n-1}]$-submodule of $\mathscr{F}(E)[\Sigma_{2n-1}]$. Then we have always $\mathscr{F}(E)(m) = 0$ if $m \neq k(n-1)+1$.

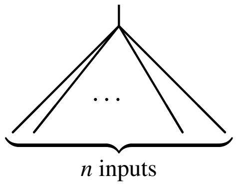
Examples

(1) If E is generated by an n-ary operation of degree 0 with no symmetries, we get that $\mathscr{F}(E)(k(n-1)+1)$ consists, as a vector space, of "parenthesized products" of $k(n-1)+1$ variables indexed by $\{1,\cdots,m=k(n-1)+1\}$. For instance, a basis of $\mathscr{F}(E)(n)$ is given by $(x_1\cdots x_n)$ and all their permutations, a basis of $\mathscr{F}(E)(2(n-1)+1)$ is given by

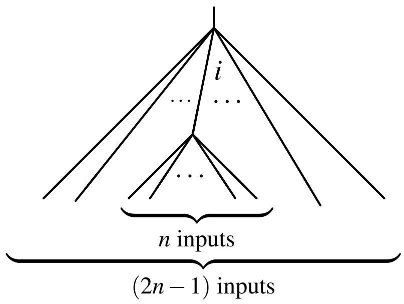
$$((x_1\cdots x_n)x_{n+1}\cdots x_{2n-1}),(x_1(x_2\cdots x_{n+1})x_{n+2}\cdots x_{2n-1}),\cdots,(x_1\cdots x_{n-1}(x_n\cdots x_{2n-1}))$$

and all their permutations.

(2) Consider that E is generated by an n-ary operation μ of degree 1. We can visualize μ by



and $\mu(Id_{i-1}\otimes\mu\otimes Id_{n-i})$ by

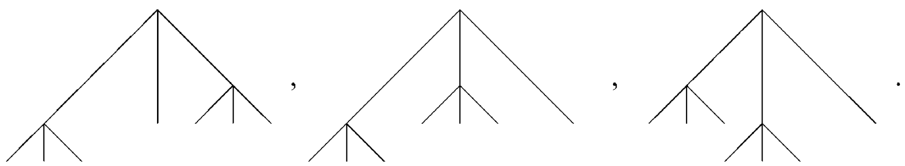


and more generally an element of $\mathscr{F}(E)(k(n-1)+1)$ by a rooted planar n-ary tree with levels and $(k(n-1)+1))$ leaves.

For instance, for $n=3$,

$$\mu\circ(Id_2\otimes\mu)\circ(\mu\otimes Id_4),\ \mu\circ(Id\otimes\mu\otimes Id)\circ(\mu\otimes Id_4),\ \mu\circ(\mu\otimes Id_2)\circ(Id_3\otimes\mu\otimes Id)$$

corresponds respectively to the following ternary trees with levels and 7 leaves:



The relations that we consider will be quadratic in the sense that we compose two n-ary multiplications. So R is a $\mathbb{K}[\Sigma_{2n-1}]$-submodule of $\mathscr{F}(E)(2n-1)$. The operad $\mathscr{R}=(R)$ is the operadic ideal generated by R. In particular, $\mathscr{R}(m)=0$ for $m=1$ and $m\neq k(n-1)+1$. In other words, $\mathscr{R}(k(n-1)+1)$ consists of all relations in $\mathscr{F}(E)(k(n-1)+1)$ induced by the relations R.

Let us focus on quadratic operads in the ternary case.

Definition 1. *Let E be a $\mathbb{K}[\Sigma_3]$-module and $\mathscr{F}(E)$ the free operad over E. If R is a $\mathbb{K}[\Sigma_5]$-submodule of $\mathscr{F}(E)(5)$ and if (R) is the operadic ideal generated by R, then the ternary quadratic operad $\mathscr{P}(\mathbb{K},E,R)$ generated by E and R is the quotient*

$$\mathscr{F}(E)/(R),$$

that is $\mathscr{P}(\mathbb{K},E,R)(m) = \mathscr{F}(E)(m)/(R)(m)$.

Note that $\mathscr{F}(E)(m) = 0$ when m is even.

Examples

Consider that E is generated by a 3-ary operation of degree 0 with no symmetries: $E \simeq \mathbb{K}[\Sigma_3]$. Let $R_{p\mathscr{A}ss_0^3}$ be the $\mathbb{K}[\Sigma_5]$-submodule of $\mathscr{F}(E)(5)$ generated by the vector

$$(x_1x_2x_3)x_4x_5 + x_1(x_2x_3x_4)x_5 + x_1x_2(x_3x_4x_5).$$

Then the corresponding quadratic operad is the operad $p\mathscr{A}ss_0^3$ for partially associative 3-ary algebras

$$p\mathscr{A}ss_0^3 = \mathscr{F}(E)/(R_{p\mathscr{A}ss_0^3}).$$

If $R_{t\mathscr{A}ss_0^3}$ is the $\mathbb{K}[\Sigma_5]$-submodule of $\mathscr{F}(E)(5)$ generated by the vectors

$$\begin{cases} (x_1x_2x_3)x_4x_5 - x_1(x_2x_3x_4)x_5, \\ x_1(x_2x_3x_4)x_5 - x_1x_2(x_3x_4x_5), \end{cases}$$

then the corresponding quadratic operad is the operad $t\mathscr{A}ss_0^3$ for totally associative 3-ary algebras. Recall that the multiplication of a ternary totally associative algebra satisfies

$$\mu \circ (\mu \otimes I_2) = \mu \circ (I_1 \otimes \mu \otimes I_1) = \mu \circ (I_2 \otimes \mu) = 0.$$

We know that for any operad $\mathscr{P}$, the spaces $\mathscr{P}(m)$ are related to the free $\mathscr{P}$-algebras. In [7] we studied the *free partially associative* 3*-ary algebras*

$$\mathscr{L}_{3pa\mathscr{A}ss}(V) = \bigoplus_{k\geq 0} \mathscr{L}^{2k+1}_{3pa\mathscr{A}ss}(V)$$

over a vector space V. We computed the dimensions of its homogeneous components and found a basis and a systematic method to write this basis. In particular, we have, if $\dim V = 1$,

$$\dim \mathscr{L}^3_{p\mathscr{A}ss_0^3}(V) = 1,\ \dim \mathscr{L}^5_{p\mathscr{A}ss_0^3}(V) = 2,\ \dim \mathscr{L}^7_{p\mathscr{A}ss_0^3}(V) = 4,$$

$$\dim \mathscr{L}^9_{p\mathscr{A}ss_0^3}(V) = 5,\ \dim \mathscr{L}^{11}_{p\mathscr{A}ss_0^3}(V) = 6,\ \dim \mathscr{L}^{13}_{p\mathscr{A}ss_0^3}(V) = 7.$$

We deduce that

$$\dim(p\mathscr{A}ss_0^3)(3) = \dim \mathbb{K}[\Sigma_3] = 6,\ \dim(p\mathscr{A}ss_0^3)(5) = 2 \times \dim \mathbb{K}[\Sigma_5] = 240$$

and more generally

$$\dim(p\mathscr{A}ss_0^3)(2k+1) = (k+1)\dim \mathbb{K}[\Sigma_{2k+1}].$$

Recall that the generating map of an operad $\mathscr{P}$ is a power series also called Poincaré series

$$g_{\mathscr{P}}(x) = \sum_{n=1}^{\infty} \chi(\mathscr{P}(n)) \frac{x^n}{n!},$$

where $\chi(\mathscr{P}(n))$ denotes the Euler characteristic of the graded vector space $\mathscr{P}(n)$. The Poincaré series of $p\mathscr{A}ss_0^3$ is then written

$$g_{p\mathscr{A}ss_0^3}(x) = \sum_{n=1}^{\infty} \dim(p\mathscr{A}ss_0^3)(n)\frac{x^n}{n!}$$

with the convention $\dim(p\mathscr{A}ss_0^3)(1) = 1$. Then

$$g_{p\mathscr{A}ss_0^3}(x) = x + x^3 + 2x^5 + 4x^7 + 5x^9 + 6x^{11} + 7x^{13} + \cdots .$$

Recall that a quadratic operad $\mathscr{P} = \mathscr{P}(\mathbb{K}, E, R)$ is Koszul if and only if, for any vector space M, the $\mathscr{P}$-algebra homology (also called the operadic homology) of the free $\mathscr{P}$-algebra $F_{\mathscr{P}}(V)$ generated by V equals V in degree 1 and vanishes in all other degrees. If $\mathscr{P}$ is a Koszul operad, then its Poincaré series $g_{\mathscr{P}}(x)$ satisfies

$$g_{\mathscr{P}}(-g_{\mathscr{P}^!}(-x)) = x,$$

where $\mathscr{P}^!$ is the dual operad. Consider $g_{p\mathscr{A}ss_0^3}$ the generating function of $p\mathscr{A}ss_0^3$ and s the formal power series satisfying

$$g_{p\mathscr{A}ss_0^3}(-s(-x)) = x.$$

We find

$$s(x) = x - x^3 + x^5 - 19x^{11} + O[x]^{12}.$$

Such a series cannot be, because of the minus signs, a Poincaré series of a quadratic operad corresponding to a multiplication of degree 0. We also conclude that the operads $p\mathscr{A}ss_0^3$ and $t\mathscr{A}ss_0^3$ with generating operation of degree 0 cannot be both Koszul and dual of each other. In the next section we compute the (graded) dual operad of $p\mathscr{A}ss_0^3$ and we will see that its generating function is a polynomial of degree 5, so (1) implies the non-Koszulity of $p\mathscr{A}ss_0^3$.

Remark. If we consider the operad $t\mathscr{A}ss_0^3$ associated to ternary totally associative algebras with operation placed in degree 0 that satisfies

$$\mu \circ (\mu \otimes I_2) = \mu \circ (I_1 \otimes \mu \otimes I_1) = \mu \circ (I_2 \otimes \mu) = 0,$$

we get the generating function

$$g_{t\mathscr{A}ss_0^3}(x) = x + x^3 + x^5 + x^7 + x^9 + \cdots + x^{2k+1} + \cdots .$$

If we suppose this operad $\mathscr{P} = t\mathscr{A}ss_0^3$ to be Koszul, we will get that the dual operad $\mathscr{P}^!$ has as generating function $g_{\mathscr{P}^!}$ satisfying $g_{\mathscr{P}}(-g_{\mathscr{P}^!}(-x)) = x$. But the identity $g_{t\mathscr{A}ss_0^3}(-h(-x)) = x$ yields to

$$h(x) = x + \sum_{m\geq 2} \frac{a_m}{m!}x^m$$

and a_n do not correspond to the dimensions of the operad $p\mathscr{A}ss_0^3$.

We get the same result for $p\mathscr{A}ss_0^{2p+1}$ associated to $(2p+1)$-ary totally associative algebras with operation placed in degree 0 that satisfies

$$\mu \circ (\mu \otimes I_{2p}) = \mu \circ (I_1 \otimes \mu \otimes I_{2p-1}) = \cdots = \mu \circ (I_{2p} \otimes \mu) = 0;$$

the generating function is

$$g_{p\mathscr{A}ss_0^{2p+1}}(x) = x + x^{2p+1} + x^{4p+1} + \cdots + x^{2kp+1} + \cdots .$$

The identity $g_{t\mathscr{A}ss_0^{2p+1}}(-h(-x)) = x$ yields to

$$h(x) = x + \sum_{m\geq 2} \frac{b_m}{m!}x^m,$$

but, if $p < 4$, the identity implies that at least one b_m is negative so the function h does not correspond to a generating function of an operad associated to a product of degree 0.

3. THE DUAL OPERAD

To compute the dual operad of the operad associated to n-ary algebras we need some differential **graded** operad. Recall that if $\mathscr{C}$ is a monoidal category, a Σ-module A is a sequence of objects $\{A(n)\}_{n\geq 1}$ in $\mathscr{C}$ with a right-Σ_n-action on $A(n)$. Then an operad in a (strict) symmetric monoidal category $\mathscr{C}$ is a Σ-module $\mathscr{P}$ together with a family of structural morphisms satisfying some associativity, equivariance, and unit conditions.

We use the fact that *dgVect*, the category of differential graded vector spaces over the base field $\mathbb{K}$ (an object of *dgVect* is a graded vector space together with a linear map d (differential) of degree 1 such that $d^2 = 0$; morphisms are linear maps preserving gradings and differentials), is a symmetric monoidal category and we can consider an operad in this category *dgVect*.

A differential **graded** operad (or dg operad) is a differential graded Σ-module with an operad structure for which the operad structure maps are differential graded morphisms.

A (nongraded) operad can be seen as a differential graded operad having trivial differentials, and nongraded objects are objects trivially graded.

Since a *dg* operad $\mathscr{P}$ is itself a monoid in a symmetric monoidal category, the bar construction applies to $\mathscr{P}$, producing a *dg* operad $\mathscr{B}(\mathscr{P})$. The linear dual of $\mathscr{B}(\mathscr{P})$ is a *dg* operad denoted by $\mathscr{C}(\mathscr{P})$ and called the cobar complex of the operad $\mathscr{P}$. We also need the dual *dg* operad $\mathscr{D}(\mathscr{P})$, which is just $\mathscr{C}(\mathscr{P})$ suitably regarded. Quadratic operads are defined as having a presentation with generators and relations and for them the dual operad will also be quadratic. For quadratic operads there is a natural transformation of functors from $\mathscr{D}(\mathscr{P})$ to $\mathscr{P}^!$ which is a quasi-isomorphism if $\mathscr{P}$ is Koszul. This concept is similar to the concept of quadratic dual and Koszulity for associative algebras.

Any quadratic operad $\mathscr{P}$ generated by a binary multiplication admits a dual operad which is also quadratic and denoted $\mathscr{P}^!$. To define it, we need to recall some definitions and notations. Let E be a Σ-module. The dual Σ-module $E^\# = \{E^\#(n)\}_{n\geq 1}$ is defined by

$$E^\#(n) = Hom_{\mathbb{K}}(E(n), \mathbb{K})$$

and the Σ_n representation on $E(n)$ determines a dual representation on $E^\#(n)$ by

$$(\lambda \cdot \sigma, \mu) := (\lambda, \mu \cdot \sigma^{-1})$$

for $\mu \in E(n), \lambda \in E^\#(n)$, and $\sigma \in \Sigma_n$. The Czech dual is the Σ-module $E^\vee = \{E^\vee(n)\}_{n\geq 1}$ with

$$E^\vee(n) = E^\#(n) \otimes sgn_n.$$

Consider the Σ-module $\widetilde{E} = \{\widetilde{E}(n)\}_{n\geq 1}$ with

$$\widetilde{E}(n) = \uparrow^{n-2} E^\#(n) \otimes sgn_n,$$

where $\uparrow^{n-2}$ denotes the suspension iterated $n-2$ times. So $\widetilde{E} = \downarrow (sE^\#)$. Then the quadratic dual operad is defined as a quotient of the free operad $\mathscr{F}(\widetilde{E})$ by relations orthogonal (in some sense) to the relations defining the original quadratic operad $\mathscr{P}$. So if $\mathscr{Q} = \mathscr{P}(E,R)$ (i.e. E corresponds to the generators and R to the relations), the dual operad $\mathscr{Q}^!$ is defined by $\mathscr{Q}^! = \mathscr{P}(\widetilde{E}, R^\perp)$ where $R^\perp \subseteq \mathscr{F}(\widetilde{E})[\Sigma_{2n-1}]$ is the annihilator with respect to some paring of the relations $R \subseteq \mathscr{F}(E)[\Sigma_{2n-1}]$ defining $\mathscr{Q}$. But notice that in general the definition of quadratic dual operad contains a suspension. If we consider a quadratic operad generated by an operation $E =< \mu > \in \mathbb{K}[\Sigma_2]$ where μ is binary of degree 0, the dual operad is still a quadratic operad generated by an operation of degree 0.

Now if we consider n-ary algebras, we have seen that we can still define the notion of quadratic operad, that is we consider a generating multiplication which is an n-ary multiplication μ, that is $E =< \mu > \subset \mathbb{K}[\Sigma_n]$, and relations which are quadratic, that is R is a $\mathbb{K}[\Sigma_{2n-1}]$-submodule. We can still define a pairing as in the case of a binary operation but now it is a map

$$<,>: \mathscr{F}(E)(n) \otimes \mathscr{F}(\widetilde{E})(n) \to \mathbb{K}.$$

Then we get $R^{\perp}$ but the dual operad $\mathscr{P}^!$ is

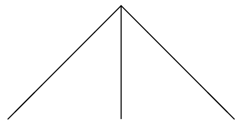
$$\mathscr{P}^!(n) = \downarrow \mathscr{F}(sE^{\#})(n)/(R^{\perp})(n) = \mathscr{F}(\widetilde{E})(n)/(R^{\perp})(n).$$

When the generating operation of $\mathscr{P}$ is of even arity (i.e. $n = 2p$), the generating operation of $\mathscr{P}^!$ is of even arity. But if the arity is odd, the dual operad is a quadratic operad with a *generating operation of odd degree*.

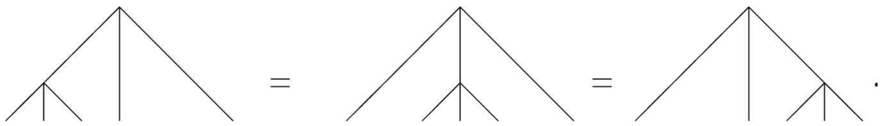
Now let us come back to the determination of the dual operad of $p\mathscr{A}ss^3_0$.

Theorem 2 [8]**.** *The dual operad of the $p\mathscr{A}ss^3_0$ operad is the operad of totally associative algebras with the operation of degree* 1, *that is $t\mathscr{A}ss^3_1$.*

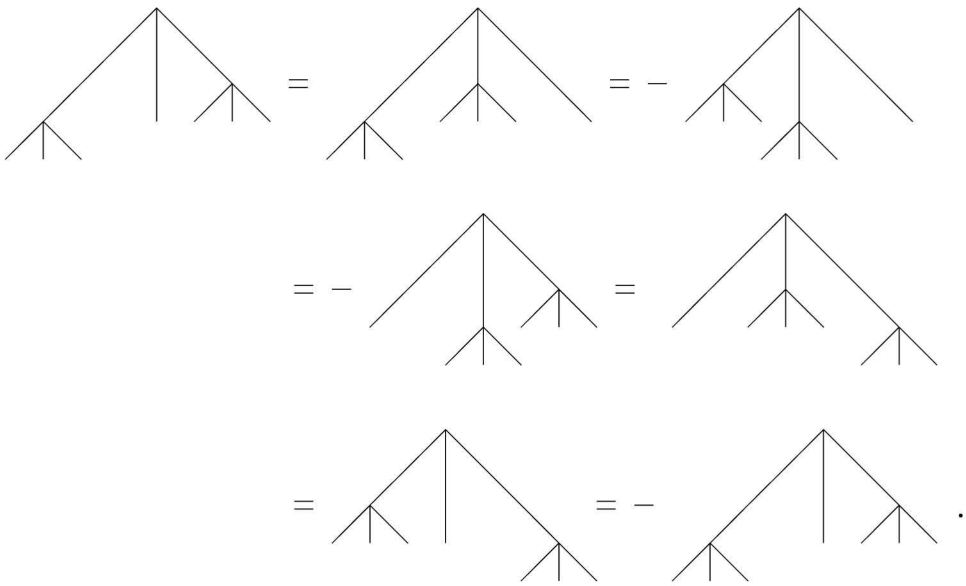
This follows directly from the definition of the dual operad of a quadratic n-ary operad. Explicitly, if $\mathscr{P} = p\mathscr{A}ss^3_0$, its dual $\mathscr{P}^!$ is generated by a ternary multiplication λ of degree 1 which can be depicted as



and the relations of R are generated by $\lambda \circ_1 \lambda = \lambda \circ_2 \lambda = \lambda \circ_3 \lambda$, which can be visualized by



We get $\dim \mathscr{P}^!(5) = 1 \times \dim \mathbb{K}[\Sigma_5] = 120$. Then all operations in $\mathscr{P}^!(kn+1)$ are trivial for $k > 3$. In fact, the operation λ is of degree 1, so we get



With the same trick we get any 7-leaves tree with level zero and $\mathscr{P}^!(7) = 0$. But any 9-leaves tree with level is obtained from a 7-leaves tree with level, so it is also zero and $\mathscr{P}^!(9) = 0$. More generally

$$\mathscr{P}^!(2k+1) = 0,$$

for any $k \geq 3$. So

$$\dim((p\mathscr{A}ss_0^3)^!)(3) = \dim((p\mathscr{A}ss_0^3)^!)(5) = 1$$

and

$$\dim((p\mathscr{A}ss_0^3)^!)(2k+1) = 0$$

for $k \geq 3$. We deduce

Theorem 3. *The generating function of the dual operad of $p\mathscr{A}ss_0^3$ is*

$$g_{(p\mathscr{A}ss_0^3)^!}(x) = x - x^3 + x^5.$$

But we saw in the previous section that the formal series satisfying

$$g_{(p\mathscr{A}ss_0^3)^!}(-s(-x)) = x$$

is of the form

$$s(x) = x - x^3 + x^5 - 19x^{11} + O[x]^{12}.$$

Then it does not correspond to the generating function of the dual of $p\mathscr{A}ss_0^3$.

Corollary 4. *The quadratic operad $p\mathscr{A}ss_0^3$ is non Koszul.*

We get the same result for the operad $p\mathscr{A}ss_0^{2p+1}$, i.e. the operad for $(2p+1)$-ary partially associative algebras with generating operation of degree 0. Its dual is the operad for $(2p+1)$-ary totally associative algebras with generating operation of degree 1. Both operads are non Koszul for $p < 4$ [8].

Theorem 5. *The generating function of the dual operad of $p\mathscr{A}ss_0^{2p+1}$ which is isomorphic to the operad $t\mathscr{A}ss_1^{2p+1}$ for $(2p+1)$-totally associative algebras with operation of degree 1 is*

$$g_{(p\mathscr{A}ss_0^{2p+1})^!}(x) = x - x^{2p+1} + x^{4p+1}.$$

Remark. In [5] and [6], we also determined natural binary operads which are not Koszul. But in this case, the multiplication is binary and the notion of degree is superfluous.

4. THE QUADRATIC OPERAD $\widetilde{p\mathscr{A}ss_0^3}$

In [10] we defined, given a quadratic operad $\mathscr{P}$, a quadratic operad $\widetilde{\mathscr{P}}$ with the following property:
For every $\mathscr{P}$-algebra A and every $\widetilde{\mathscr{P}}$-algebra B, the tensor product $A \otimes B$ is a $\mathscr{P}$-algebra.

Proposition 6. *We have*

$$\widetilde{p\mathscr{A}ss_0^3} = t\mathscr{A}ss_0^3.$$

Here the product is considered to be of degree 0. In many classical cases, we have seen that $\mathscr{P}^! = \widetilde{\mathscr{P}}$. Some examples where this equality is not realized are constructed using binary non-associative algebras.

Remark: The operad of Jordan Triple systems. A Jordan Triple system on a vector space is a 3-ary product μ satisfying the commutativity condition

$$\mu(x_1, x_2, x_3) = \mu(x_3, x_2, x_1)$$

and

$$\mu(x_1,x_2,\mu(x_3,x_4,x_5))+\mu(x_3,\mu(x_2,x_1,x_4),x_5)=\mu(\mu(x_1,x_2,x_3),x_4,x_5)$$
$$+\mu(x_3,x_4,\mu(x_1,x_2,x_5)).$$

We denote by $\mathscr{J}ord_3$ the operad of algebras defined by a Jordan Triple system. In [4] it is proved that this quadratic operad is Koszul computing its dual. But the definition of the dual is based on the nongraded version of [3]. Then they consider that the dual corresponds to *papas* for partially associative and partially skewsymmetric (i.e. $(x_1,x_2,x_3)=-(x_3,x_2,x_1)$ where $(\ , \ , \ )$ is the associator) algebras but with generating operation of degree 0 and we are not sure that the Koszulity of $\mathscr{J}ord_3$ is satisfied. This calculus using product of degree 1 is being worked out by Nicolas Goze and myself.

ACKNOWLEDGEMENTS

We thank in particular Eugen Paal and Sergei Silvestrov and organizers of the Algebra, Geometry, and Mathematical Physics Workshop for allowing us to present this work.

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Koszuli tingimuse rikkumine ternaarses osaliselt assotsiatiivses operaadis

Elisabeth Remm

On tõestatud, et ternaarsete osaliselt assotsiatiivsete algebrate operaadis Koszuli tingimus ei kehti. Eesmärgiks on rõhutada duaalse operaadi arvutamise probleemi *n*-aarsete algebrate ruutoperaadi korral, eriti kui *n* on paaritu. Õigupoolest defineeritakse duaalne operaad gradueeritud (diferentsiaal-) operaadi formalismis. Koszuli tingimuse rikkumine laieneb ka teistele osaliselt assotsiatiivsete $(2p+1)$-aarsete algebrate operaadidele, kuigi osaliselt assotsiatiivsete $(2p)$-aarsete algebrate operaadides Koszuli tingimus kehtib.